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Boundary reflection matrix for ade affine Toda field theory

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Abstract. We present a complete set of conjectures for the exact boundary reflection matrix for ade affine Toda field theory defined on a half line with the Neumann boundary condition.

1. Introduction

About a decade ago, studies on the integrable quantum field theory defined on a half line ($-\infty < x \leqslant 0$) were initiated using symmetry principles under the assumption that the quantum integrability of the model remains intact [1]. The boundary Yang–Baxter equation, unitarity relation for the boundary reflection matrix $K_a^b(\theta)$ which was conceived to describe the scattering process off a wall was introduced [1].

Recently, the boundary crossing unitarity relations [2] and the boundary bootstrap equations [3] were introduced. Subsequently, a variety of solutions of the algebraic equations for the affine Toda field theory have been constructed [2–6]. However, a proper interpretation of these solutions in terms of the Lagrangian quantum field theory had been unknown.

On the other hand, non-trivial boundary potentials which do not destroy the integrability properties in the sense that there still exist an infinite number of conserved currents has been determined [2, 6–10]. The stability problem of certain models with boundary potential has also been discussed [6, 11].

Very recently, we have proposed a formalism [12] to compute a boundary reflection matrix in the framework of the Lagrangian quantum field theory with a boundary [13–15]. The idea is to extract the boundary reflection matrix directly from the two-point correlation function in the coordinate space. And it has revealed a number of striking features of the perturbative quantum field theory defined on a half line.

Using this formalism, we determined the exact boundary reflection matrix for the sinh–Gordon model ($a_1^{(1)}$ affine Toda theory) and Bullough–Dodd model ($a_2^{(2)}$ affine Toda theory) with the Neumann boundary condition [12]. If we assume the strong–weak coupling 'duality', these solutions are unique.

The above two models have a particle spectrum with only one mass. On the other hand, when the theory has a particle spectrum with more than one mass, each one-loop contribution from different types of Feynman diagrams has a variety of non-meromorphic terms. We expect actual cancellation of these non-meromorphic terms ought to be essential for a boundary reflection matrix to have a nice analytic property.

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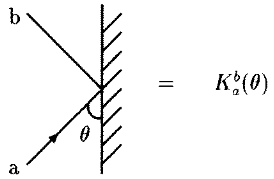

Figure 1. Boundary reflection matrix.

In [16], we evaluated the one-loop boundary reflection matrix for $d_4^{(1)}$ affine Toda field theory and showed a remarkable cancellation of non-meromorphic terms among themselves. This result also enabled us to determine the exact boundary reflection matrix uniquely under the assumption of the strong–weak coupling 'duality'. It turned out that the boundary reflection matrix has singularities which can be accounted for by a new type of singularities of Feynman diagrams for a theory defined on a half line.

In this paper, we present a complete set of conjectures for the exact boundary reflection matrix for ade affine Toda field theory defined on a half line with the Neumann boundary condition. With this boundary condition, we expect the strong–weak coupling 'duality', which is a symmetry of the model defined on a full line, to still be effective.

In section 2, we review the formalism developed in [12]. Particularly, we give a more informative form of the formulae given in [12]. In section 3, we present a complete set of conjectures for the exact boundary reflection matrix for ade affine Toda field theory with the Neumann boundary condition. Finally, we make conclusions in section 4. In an appendix, we present the one-loop result as well as the complete set of solutions of the boundary bootstrap equations for $a_3^{(1)}$ theory.

2. Boundary reflection matrix

The action for affine Toda field theory defined on a half line ($-\infty < x \leqslant 0$) is given by

$$S(\Phi) = \int_{-\infty}^{0} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}t \left(\frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i \, \mathrm{e}^{\beta \alpha_i \cdot \Phi} \right) \tag{1}$$

where

$$\alpha_0 = -\sum_{i=1}^{r} n_i \alpha_i \qquad \text{and} \qquad n_0 = 1\,.$$

The field ϕ^a ($a = 1, \ldots, r$) is the ath component of the scalar field Φ and α_i ($i = 1, \ldots, r$) are simple roots of a Lie algebra g with rank r normalized so that the universal function $B(\beta)$ through which the dimensionless coupling constant β appears in the S-matrix takes the following form:

$$B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{(1 + \beta^2/4\pi)}\,. \tag{2}$$

The m sets the mass scale and the n_i are the so-called Kac labels which are characteristic integers defined for each Lie algebra. The quantity $h = \sum_0^r n_i$ is called the Coxeter number.

Here we consider the model with no boundary potential, which corresponds to the Neumann boundary condition: $\frac{\partial \phi^a}{\partial x} = 0$ at $x = 0$. This case is believed to be quantum stable in the sense that the existence of a boundary does not change the structure of the quantum spectrum determined for the same theory defined on a full line.

In classical field theory, it is quite clear how we extract the boundary reflection matrix. It is the coefficient of the reflection term in the classical two-point correlation function, namely it is 1:

$$G_N(t',x';t,x) = G(t',x';t,x) + G(t',x';t,-x) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{i}}{p^2 - m_a^2 + \mathrm{i}\varepsilon} \, \mathrm{e}^{-\mathrm{i}w(t'-t)} (\mathrm{e}^{\mathrm{i}k(x'-x)} + \mathrm{e}^{\mathrm{i}k(x'+x)}) \, . \tag{3}$$

We may use the k-integrated version:

$$G_N(t',x';t,x) = \int \frac{\mathrm{d}w}{2\pi} \frac{1}{2\bar{k}} \, \mathrm{e}^{-\mathrm{i}w(t'-t)} (\mathrm{e}^{\mathrm{i}\bar{k}|x'-x|} + \mathrm{e}^{-\mathrm{i}\bar{k}(x'+x)}) \qquad \bar{k} = \sqrt{w^2 - m_a^2} \, . \tag{4}$$

We find that the unintegrated version is very useful to extract the asymptotic part of the two-point correlation function far away from the boundary.

In quantum field theory, it also seems quite natural to extend the above idea in order to extract the quantum boundary reflection matrix directly from the quantum two-point correlation function. This idea has been pursued in [12] to extract the one-loop boundary reflection matrix.

To compute two-point correlation functions at one-loop order, we follow the idea of the conventional perturbation theory [13–15]. That is, we generate relevant Feynman diagrams and then evaluate each of them by using the zeroth-order two-point function for each line occurring in the Feynman diagrams.

At one-loop order, there are three types of Feynman diagram contributing to the two-point correlation function as depicted in figure 2.

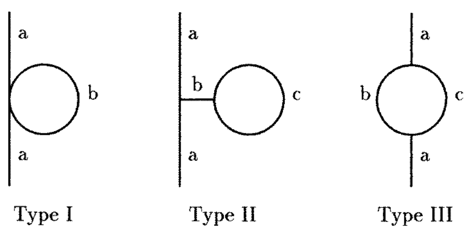

Figure 2. Diagrams for the one-loop two-point function.

For a theory defined on a full line which has translational symmetry in space and time directions, type I, II diagrams have a logarithmic infinity independent of the external energy–momenta and are the only divergent diagrams in $1+1$ dimensions. This infinity is usually absorbed into the infinite-mass renormalization. Type III diagrams have finite corrections depending on the external energy–momenta and produces a double pole to the two-point correlation function.

The remedy for these double poles is to introduce a counterterm to the original Lagrangian to cancel this term (or to renormalize the mass). In addition, to maintain the residue of the pole, we have to introduce wavefunction renormalization. Then the renormalized two-point correlation function remains the same as the tree level one with renormalized mass m_a, whose ratios are the same as the classical value. This mass renormalization procedure can be generalized to arbitrary order of loops.

Now let us consider each diagram for a theory defined on a half line. Type I diagram gives the following contribution:

$$\int_{-\infty}^{0} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}t_1 \, G_N(t,x;t_1,x_1) \, G_N(t',x';t_1,x_1) \, G_N(t_1,x_1;t_1,x_1) \, . \tag{5}$$

From type II diagram, we can read off the following expression:

$$\int_{-\infty}^{0} \mathrm{d}x_1\, \mathrm{d}x_2 \int_{-\infty}^{\infty} \mathrm{d}t_1\, \mathrm{d}t_2\, G_N(t, x; t_1, x_1)\; G_N(t', x'; t_1, x_1)\, G_N(t_1, x_1; t_2, x_2)\, G_N(t_2, x_2; t_2, x_2)\,. \tag{6}$$

Type III diagram gives the following contribution:

$$\int_{-\infty}^{0} \mathrm{d}x_1\, \mathrm{d}x_2 \int_{-\infty}^{\infty} \mathrm{d}t_1\, \mathrm{d}t_2\, G_N(t, x; t_1, x_1)\, G_N(t', x'; t_2, x_2)\, G_N(t_2, x_2; t_1, x_1)\, G_N(t_2, x_2; t_1, x_1)\,. \tag{7}$$

After the infinite as well as finite mass renormalization, the remaining terms coming from type I, II and III diagrams can be written as follows with different I_i functions [12]:

$$\int \frac{\mathrm{d}w}{2\pi} \frac{\mathrm{d}k}{2\pi} \frac{\mathrm{d}k'}{2\pi} \mathrm{e}^{-\mathrm{i}w(t'-t)}\, \mathrm{e}^{\mathrm{i}(kx+k'x')} \frac{\mathrm{i}}{w^2 - k^2 - m_a^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}}{w^2 - k'^2 - m_a^2 + \mathrm{i}\varepsilon} I_i(w, k, k')\,. \tag{8}$$

Contrary to the other terms which resemble those of a full line, this integral has two spatial momentum integrations.

In the asymptotic region far away from the boundary, these terms can be evaluated up to an exponentially damped term as x, x' go to $-\infty$, yielding the following result for the elastic boundary reflection matrix $K_a(\theta)$ defined as the coefficient of the reflected term of the two-point correlation function:

$$\int \frac{\mathrm{d}w}{2\pi}\, \mathrm{e}^{-\mathrm{i}w(t'-t)} \frac{1}{2\bar{k}} (\mathrm{e}^{\mathrm{i}\bar{k}|x'-x|} + K_a(w)\, \mathrm{e}^{-\mathrm{i}\bar{k}(x'+x)}) \qquad \bar{k} = \sqrt{w^2 - m_a^2}\,. \tag{9}$$

$K_a(\theta)$ is obtained using $w = m_a \cosh\theta$. Here we list each one-loop contribution to $K_a(\theta)$ from the three types of diagram depicted in figure 2 [12]:

$$K_a^{(\mathrm{I})}(\theta) = \frac{1}{4m_a \sinh\theta} \left(\frac{1}{2\sqrt{m_a^2 \sinh^2\theta + m_b^2}} + \frac{1}{2m_b} \right) C_1\, S_1 \tag{10}$$

$$K_a^{(\mathrm{II})}(\theta) = \frac{1}{4m_a \sinh\theta} \left(\frac{-\mathrm{i}}{(4m_a^2 \sinh^2\theta + m_b^2) 2\sqrt{m_a^2 \sinh^2\theta + m_c^2}} + \frac{-\mathrm{i}}{2m_b^2 m_c} \right) C_2\, S_2 \tag{11}$$

$$K_a^{(\mathrm{III})}(\theta) = \frac{1}{4m_a \sinh\theta} (4I_3(k_1 = 0, k_2 = \bar{k}) + 4I_3(k_1 = \bar{k}, k_2 = 0))\, C_3\, S_3\,. \tag{12}$$

We remark that the extra factor half which was reported as missing in [12] is found to arise from the delta function integral(s) of the spatial loop momentum(a). That is, $\int \mathrm{d}k\, \delta(2k) = \frac{1}{2}$ instead of 1.

C_i, S_i denote numerical coupling factors and symmetry factors, respectively. I_3 is defined by

$$I_3 \equiv \frac{1}{4} \left(\frac{\mathrm{i}}{2\bar{w}_1(\bar{w}_1 - \tilde{w}_1^+)(\bar{w}_1 - \tilde{w}_1^-)} + \frac{\mathrm{i}}{(\tilde{w}_1^+ - \bar{w}_1)(\tilde{w}_1^+ + \bar{w}_1)(\tilde{w}_1^+ - \tilde{w}_1^-)} \right) \tag{13}$$

where

$$\bar{w}_1 = \sqrt{k_1^2 + m_b^2} \qquad \tilde{w}_1^+ = w + \sqrt{k_2^2 + m_c^2} \qquad \tilde{w}_1^- = w - \sqrt{k_2^2 + m_c^2}\,. \tag{14}$$

It should be remarked that this term should be symmetrized with respect to m_b, m_c with a half.

The expression for a contribution from a type III diagram can be rewritten in the following form:

$$K_a^{(\mathrm{III})} = \frac{\mathrm{i}}{4m_a \sinh\theta}\, C_3\, S_3 \Bigg(\frac{\cos\theta_{ab}^c}{4m_a m_b^2(\cosh^2\theta - \cos^2\theta_{ab}^c)} - \frac{m_a\cosh^2\theta + m_b\cos\theta_{ab}^c}{2m_a m_b^2 2\sqrt{m_a^2\sinh^2\theta + m_c^2}(\cosh^2\theta - \cos^2\theta_{ab}^c)} + \frac{\cos\theta_{ac}^b}{4m_a m_c^2(\cosh^2\theta - \cos^2\theta_{ac}^b)} - \frac{m_a\cosh^2\theta + m_c\cos\theta_{ac}^b}{2m_a m_c^2 2\sqrt{m_a^2\sinh^2\theta + m_b^2}(\cosh^2\theta - \cos^2\theta_{ac}^b)} \Bigg) \tag{15}$$

where θ_{ab}^c is a usual fusion angle defined by

$$\cos\theta_{ab}^c = \frac{m_c^2 - m_a^2 - m_b^2}{2m_a m_b}\,. \tag{16}$$

Let us note a few interesting points. Firstly, all the expressions in (10)–(12) have, in general, non-meromorphic terms when the theory has a mass spectrum with more than one mass. Cancellation of these terms is expected to occur for the boundary reflection matrix to have a nice analytic property. We have verified this non-trivial cancellation for $d_4^{(1)}$ theory in [16] and the result for $a_3^{(1)}$ theory is presented in the appendix. Secondly, the Feynman diagrams have (simple pole) singularities which are absent for the theory defined on a full line. A general study on the analytic property of the boundary reflection matrix is definitely needed, while that for the scattering matrix has been done extensively [17].

Moreover, the position of poles are directly related with fusion angles as in (15) and less obviously as in (11). Later in the appendix, we will see a non-trivial cancellation of non-meromorphic terms and the fact that the new type of singularity accounts for the singularities of the exact boundary reflection matrix.

3. The boundary reflection matrix for *ade* affine Toda theory

The exact S-matrix for integrable quantum field theory defined on a full line has been conjectured using the symmetry principles such as the Yang–Baxter equation, unitarity, crossing relation, real analyticity and bootstrap equation [18–22]. This program relies entirely on the assumed quantum integrability of the model as well as the fundamental assumptions such as strong–weak coupling 'duality' and 'minimality'.

In order to determine the exact S-matrix uniquely, Feynman's perturbation theory has been used [23–27] and shown to agree well with the conjectured 'minimal' S-matrices. In perturbation theory, the S-matrix is extracted from the four-point correlation function with the LSZ reduction formalism. The singularity structures were examined in terms of the Landau singularity [17], of which odd order poles are interpreted as coming from the intermediate bound states.

In determining the whole set of scattering matrix elements, it is essentially sufficient to determine the element for the so-called 'elementary particle'. Starting from that element, we can determine all the other elements using the bootstrap equations [20]. This is also true for the boundary reflection matrix. In $a_n^{(1)}$ theory, the 'elementary particle' is the lightest one corresponding to two end points of the Dynkin diagram. In $d_n^{(1)}$ theory, 'elementary particles' are those corresponding to (anti-)spinor representations. In $e_6^{(1)}$ theory, 'elementary particles' are the lightest ones which are conjugate to each other corresponding to two end points of the Dynkin diagram. In $e_7^{(1)}$ and $e_8^{(1)}$ theories, it is the lightest one corresponding to the end point of the longer arm of the Dynkin diagram.

Figure 3. Dynkin diagram for a_n.

Let us start from $a_n^{(1)} (n \geqslant 1)$ theory. The boundary reflection matrix for the 'elementary particles' can be coded into the following pyramid of exponents of the factors $[x]$ which appear in the boundary reflection matrix.

1
1 1
1 1 1
1 1 1 1
1 1 1 1 1

Figure 4. A pyramid of exponents for a_n theory.

It means that

$$K_1(\theta) = K_n(\theta) = \prod_{k=1,\text{step }2}^{2h-3} [k/2] \tag{17}$$

where

$$[x] = \frac{(x-1/2)(x+1/2)}{(x-1/2+B/2)(x+1/2-B/2)} \qquad (x) = \frac{\sinh(\theta/2 + \mathrm{i}\pi x/2h)}{\sinh(\theta/2 - \mathrm{i}\pi x/2h)} . \tag{18}$$

From these elements of the boundary reflection matrix, we can, in principle, determine all the other elements using the boundary bootstrap equations.

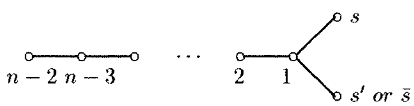

Figure 5. Dynkin diagram for d_n.

For $d_n^{(1)} (n \geqslant 2)$ theory, a pyramid of exponents takes a slightly complicated form. $d_2^{(1)}$ theory is equal to two copies of sinh–Gordon theory which is $a_1^{(1)}$ theory and $d_3^{(1)}$ theory is equal to $a_3^{(1)}$ theory.

1
1 1 1
1 1 2 1 1
1 1 2 2 2 1 1
1 1 2 2 3 2 2 1 1

Figure 6. The first pyramid of exponents for d_n theory.

It means that

$$K_s(\theta) = K_{s'(\bar{s})}(\theta) = \prod_{k=1,\text{step }2}^{2h-3} [k/2]^{x_k} \tag{19}$$

where x_k are the exponents in sequence from (to) left to (from) right in figure 6. The rule of figure 6 is the following. At odd rows except the apex, prepare two copies of the middle number and put them on two sites neighbouring the centre, pushing the others away towards both sides and increment the original middle number by one unit. At even rows, do the same thing as for odd rows but leave the middle number without incrementing it. From these elements of the boundary reflection matrix, we can determine all the other elements.

On the other hand, a pyramid of exponents for the lightest particle corresponding to the end point of the longer arm of the Dynkin diagram take the following form.

1
1 2 1
1 1 2 1 1
1 1 1 2 1 1 1
1 1 1 1 2 1 1 1 1

Figure 7. The second pyramid of exponents for d_n theory.

It means that

$$K_{n-2}(\theta) = \prod_{k=1,\text{step }2}^{2h-3} [k/2]^{x_k} \tag{20}$$

where x_k are the exponents in figure 7. The rule of the figure 7 is that only the middle number is two except the apex. From these data, we cannot determine all the other elements for each $d_n^{(1)}$ theory. However, it obviously looks simpler than the elements corresponding to (anti-)spinor representations.

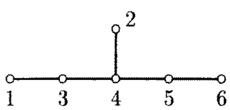

Figure 8. Dynkin diagram for e_6.

For $e_n^{(1)}$ theory, we have checked the conjectured boundary reflection matrices of the 'elementary particles' by perturbation theory. Other elements for particles which are not 'elementary' are determined using the boundary bootstrap equations.

For $e_6^{(1)}$ theory ($h = 12$), a complete list is

$$\begin{aligned}
K_1(\theta) &= [1/2][3/2][5/2][7/2]^2[9/2]^2[11/2]^2[13/2]^2[15/2]^2[17/2][19/2][21/2] \\
K_2(\theta) &= [1/2][3/2][5/2]^2[7/2]^3[9/2]^3[11/2]^3[13/2]^2[15/2]^3[17/2]^2[19/2][21/2] \\
K_3(\theta) &= [1/2][3/2]^2[5/2]^3[7/2]^4[9/2]^4[11/2]^4[13/2]^4[15/2]^3[17/2]^2[19/2]^2[21/2] \\
K_4(\theta) &= [1/2][3/2]^3[5/2]^5[7/2]^6[9/2]^6[11/2]^6[13/2]^5[15/2]^4[17/2]^3[19/2]^2[21/2] \\
K_5(\theta) &= K_3(\theta) \\
K_6(\theta) &= K_1(\theta)\,.
\end{aligned} \tag{21}$$

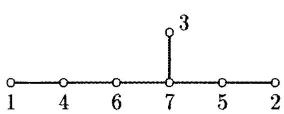

Figure 9. Dynkin diagram for e_7.

For $e_7^{(1)}$ theory ($h = 18$), a complete list is

$$\begin{aligned}
K_1(\theta) &= [1/2][3/2][5/2][7/2][9/2]^2[11/2]^2[13/2]^2[15/2]^2[17/2]^3[19/2]^2 \\
&\quad \times[21/2]^2[23/2]^2[25/2]^2[27/2][29/2][31/2][33/2] \\
K_2(\theta) &= [1/2][3/2][5/2][7/2]^2[9/2]^2[11/2]^3[13/2]^3[15/2]^3[17/2]^3[19/2]^2 \\
&\quad \times[21/2]^3[23/2]^3[25/2]^2[27/2]^2[29/2][31/2][33/2] \\
K_3(\theta) &= [1/2][3/2][5/2]^2[7/2]^3[9/2]^4[11/2]^4[13/2]^4[15/2]^4[17/2]^5[19/2]^4 \\
&\quad \times[21/2]^4[23/2]^3[25/2]^3[27/2]^3[29/2]^2[31/2][33/2] \\
K_4(\theta) &= [1/2][3/2]^2[5/2]^2[7/2]^3[9/2]^4[11/2]^4[13/2]^4[15/2]^5[17/2]^5[19/2]^4 \\
&\quad \times[21/2]^4[23/2]^4[25/2]^3[27/2]^2[29/2]^2[31/2]^2[33/2]
\end{aligned} \tag{22}$$

$$K_5(\theta) = [1/2][3/2]^2[5/2]^3[7/2]^4[9/2]^5[11/2]^6[13/2]^6[15/2]^6[17/2]^6[19/2]^5$$
$$\times[21/2]^5[23/2]^5[25/2]^4[27/2]^3[29/2]^2[31/2]^2[33/2]$$
$$K_6(\theta) = [1/2][3/2]^2[5/2]^4[7/2]^5[9/2]^6[11/2]^6[13/2]^7[15/2]^7[17/2]^7[19/2]^6$$
$$\times[21/2]^6[23/2]^5[25/2]^4[27/2]^3[29/2]^3[31/2]^2[33/2]$$
$$K_7(\theta) = [1/2][3/2]^3[5/2]^5[7/2]^7[9/2]^8[11/2]^9[13/2]^9[15/2]^9[17/2]^9[19/2]^8$$
$$\times[21/2]^7[23/2]^6[25/2]^5[27/2]^4[29/2]^3[31/2]^2[33/2]\,.$$

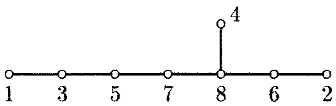

Figure 10. Dynkin diagram for e_8.

For $e_8^{(1)}$ theory($h = 30$), a complete list is

$$K_1 = [1/2][3/2][5/2][7/2][9/2][11/2]^2[13/2]^2[15/2]^2[17/2]^2[19/2]^3[21/2]^3$$
$$\times[23/2]^3[25/2]^3[27/2]^3[29/2]^3[31/2]^2[33/2]^3[35/2]^3[37/2]^3[39/2]^3$$
$$\times[41/2]^2[43/2]^2[45/2]^2[47/2]^2[49/2][51/2][53/2][55/2][57/2]$$
$$K_2 = [1/2][3/2][5/2][7/2]^2[9/2]^2[11/2]^3[13/2]^4[15/2]^4[17/2]^4[19/2]^4[21/2]^4$$
$$\times[23/2]^5[25/2]^5[27/2]^5[29/2]^5[31/2]^4[33/2]^5[35/2]^5[37/2]^4[39/2]^4$$
$$\times[41/2]^3[43/2]^3[45/2]^4[47/2]^3[49/2]^2[51/2]^2[53/2][55/2][57/2]$$
$$K_3 = [1/2][3/2]^2[5/2]^2[7/2]^2[9/2]^3[11/2]^4[13/2]^4[15/2]^4[17/2]^5[19/2]^6[21/2]^6$$
$$\times[23/2]^6[25/2]^6[27/2]^6[29/2]^6[31/2]^5[33/2]^5[35/2]^6[37/2]^6[39/2]^5$$
$$\times[41/2]^4[43/2]^4[45/2]^4[47/2]^3[49/2]^2[51/2]^2[53/2]^2[55/2]^2[57/2]$$
$$K_4 = [1/2][3/2][5/2]^2[7/2]^3[9/2]^4[11/2]^5[13/2]^5[15/2]^6[17/2]^6[19/2]^7[21/2]^7$$
$$\times[23/2]^7[25/2]^7[27/2]^7[29/2]^8[31/2]^7[33/2]^7[35/2]^6[37/2]^6[39/2]^6$$
$$\times[41/2]^5[43/2]^5[45/2]^4[47/2]^4[49/2]^3[51/2]^3[53/2]^2[55/2][57/2]$$
$$K_5 = [1/2][3/2]^2[5/2]^3[7/2]^4[9/2]^5[11/2]^6[13/2]^6[15/2]^7[17/2]^8[19/2]^9[21/2]^9 \qquad (23)$$
$$\times[23/2]^9[25/2]^9[27/2]^9[29/2]^9[31/2]^8[33/2]^8[35/2]^8[37/2]^8[39/2]^7$$
$$\times[41/2]^6[43/2]^6[45/2]^5[47/2]^4[49/2]^3[51/2]^3[53/2]^3[55/2]^2[57/2]$$
$$K_6 = [1/2][3/2]^2[5/2]^3[7/2]^4[9/2]^5[11/2]^7[13/2]^8[15/2]^8[17/2]^9[19/2]^9[21/2]^9$$
$$\times[23/2]^{10}[25/2]^{10}[27/2]^{10}[29/2]^{10}[31/2]^9[33/2]^9[35/2]^9[37/2]^8[39/2]^7$$
$$\times[41/2]^6[43/2]^6[45/2]^6[47/2]^5[49/2]^4[51/2]^3[53/2]^2[55/2]^2[57/2]$$
$$K_7 = [1/2][3/2]^2[5/2]^4[7/2]^6[9/2]^7[11/2]^8[13/2]^9[15/2]^{10}[17/2]^{11}[19/2]^{12}[21/2]^{12}$$
$$\times[23/2]^{12}[25/2]^{12}[27/2]^{12}[29/2]^{12}[31/2]^{11}[33/2]^{11}[35/2]^{10}[37/2]^9$$
$$\times[39/2]^9[41/2]^8[43/2]^7[45/2]^6[47/2]^5[49/2]^4[51/2]^4$$
$$\times[53/2]^3[55/2]^2[57/2]$$
$$K_8 = [1/2][3/2]^3[5/2]^5[7/2]^7[9/2]^9[11/2]^{11}[13/2]^{12}[15/2]^{13}[17/2]^{14}[19/2]^{15}[21/2]^{15}$$
$$\times[23/2]^{15}[25/2]^{15}[27/2]^{15}[29/2]^{15}[31/2]^{14}[33/2]^{13}[35/2]^{12}[37/2]^{11}$$
$$\times[39/2]^{10}[41/2]^9[43/2]^8[45/2]^7[47/2]^6[49/2]^5[51/2]^4[53/2]^3$$
$$\times[55/2]^2[57/2]\,.$$

We remark that we have extensive direct proofs for these conjectures by perturbation theory which are basically case-by-case works. Parts of them have already been presented in [12, 16] and are presented in the appendix of this paper. These conjectured boundary reflection matrices are also tested against various algebraic requirements such as the boundary crossing unitarity relations and always give consistent results.

4. Conclusions

In this paper, we presented a complete set of conjectures for the exact boundary reflection matrix for ade affine Toda field theory defined on a half line with Neumann boundary condition. These conjectures are based on extensive direct proofs by perturbation theory and are tested against various algebraic requirements such as the boundary crossing unitarity relations and the boundary bootstrap equations.

Surprisingly enough, these solutions have very rich pole structures in a physical strip $(0 \leqslant \mathrm{Im}(\theta) < \pi)$. However, structures of these singularities are explainable in terms of Feynman diagrams in figure 2 which definitely have no singularity for the theory defined on a full line and their positions of poles which are produced by the Feynman diagrams are related with fusing angles for affine Toda field theory as in (15).

In the appendix, we presented a detailed computation for $a_3^{(1)}$ affine Toda field theory up to one-loop order in order to demonstrate a remarkable cancellation of non-meromorphic terms which are always present for each diagram when the model has a particle spectrum with more than one mass. Using this result, we also determined the exact boundary reflection matrix under the assumption of the strong–weak coupling 'duality', which turned out to be 'non-minimal'. We also presented the complete set of solutions of the boundary bootstrap equations.

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Appendix A. $a_3^{(1)}$ affine Toda theory

We have to fix the normalization of roots so that the standard $B(\beta)$ function takes the form in (2).

We use the Lagrangian density given by

$$\begin{aligned} V(\Phi) = {} & 2m^2\phi_1\phi_1^* + 2m^2\phi_2\phi_2 + \mathrm{i}m^2\beta\phi_1\phi_1\phi_2 - \mathrm{i}m^2\beta\phi_2\phi_1^*\phi_1^* \\ & - \tfrac{1}{24}m^2\beta^2\phi_1\phi_1\phi_1\phi_1 + \tfrac{1}{4}m^2\beta^2\phi_1\phi_1\phi_1^*\phi_1^* + m^2\beta^2\phi_1\phi_1^*\phi_2\phi_2 \\ & + \tfrac{1}{6}m^2\beta^2\phi_2\phi_2\phi_2\phi_2 - \tfrac{1}{24}m^2\beta^2\phi_1^*\phi_1^*\phi_1^*\phi_1^* + \mathrm{O}(\beta^3)\,. \end{aligned} \tag{A1}$$

The scattering matrix of this model is given by [19]

$$\begin{aligned} & S_{11}(\theta) = S_{33}(\theta) = \{1\} \qquad S_{12}(\theta) = \{2\} \qquad S_{22}(\theta) = \{1\}\{3\} \\ & \{x\} = \frac{(x-1)(x+1)}{(x-1+B)(x+1-B)}\,. \end{aligned} \tag{A2}$$

Here B is the same function defined in (2). For this model, $h = 4$ and from now on we set $m = 1$.

First we consider the light particle corresponding to ϕ_1 or its conjugate. It is understood that a suitable choice between a conjugate pair has to be made depending on a chosen direction of time flow. There are two possible configurations for a type I diagram. One is $b = \phi_1$ or its conjugate and the other is $b = \phi_2$ in the notation of figure 2. The ϕ_1 loop contribution is the following:

$$K_1(\theta)^{(I-1)} = \frac{1}{4\sqrt{2}\sinh\theta}\left(\frac{1}{2\sqrt{2}\cosh\theta} + \frac{1}{2\sqrt{2}}\right) \times \left(\frac{-\mathrm{i}}{4}\beta^2\right) \times 4\,. \tag{A3}$$

The ϕ_2 loop contribution is the following:

$$K_1(\theta)^{(I-2)} = \frac{1}{4\sqrt{2}\sinh\theta}\left(\frac{1}{2\sqrt{2\sinh^2\theta+4}} + \frac{1}{4}\right) \times (-\mathrm{i}\beta^2) \times 1\,. \tag{A4}$$

There are no configurations for the type II diagram for the $a_3^{(1)}$ model. In fact, this is the case for any $a_n^{(1)}$ theory.

For a type III diagram, there exists only one configuration with $b = \phi_1, c = \phi_2$ symmetrized. For $b = \phi_1, c = \phi_2$, when $k_1 = 0, k_2 = k$,

$$\bar{w}_1 = \sqrt{2} \qquad \tilde{w}_1^+ = \sqrt{2}\cosh\theta + \sqrt{2\sinh^2\theta + 4} \qquad \tilde{w}_1^- = \sqrt{2}\cosh\theta - \sqrt{2\sinh^2\theta+4} \tag{A5}$$

and when $k_1 = k, k_2 = 0$,

$$\bar{w}_1 = \sqrt{2\sinh^2\theta+2} \qquad \tilde{w}_1^+ = \sqrt{2}\cosh\theta + 2 \qquad \tilde{w}_1^- = \sqrt{2}\cosh\theta - 2\,. \tag{A6}$$

For $b = \phi_2, c = \phi_1$, when $k_1 = 0, k_2 = k$,

$$\bar{w}_1 = 2 \qquad \tilde{w}_1^+ = \sqrt{2}\cosh\theta + \sqrt{2\sinh^2\theta+2} \qquad \tilde{w}_1^- = \sqrt{2}\cosh\theta - \sqrt{2\sinh^2\theta+2} \tag{A7}$$

and when $k_1 = k, k_2 = 0$,

$$\bar{w}_1 = \sqrt{2\sinh^2\theta+4} \qquad \tilde{w}_1^+ = \sqrt{2}\cosh\theta + \sqrt{2} \qquad \tilde{w}_1^- = \sqrt{2}\cosh\theta - \sqrt{2}\,. \tag{A8}$$

The result for type III diagram can be obtained by inserting above data into equation (12):

$$K_1(\theta)^{(\mathrm{III})} = \frac{1}{4\sqrt{2}\sinh\theta}\left(-\frac{\mathrm{i}}{8\sqrt{2}\cosh\theta} - \frac{\mathrm{i}}{8\sqrt{2}\sqrt{\sinh^2\theta+2}} + \frac{\mathrm{i}}{16(\sqrt{2}\cosh\theta+1)}\right) \times(-\beta^2)\times 4\,. \tag{A9}$$

Adding the above contributions as well as the classical value 1, the boundary reflection matrix for the light particle is given by

$$K_1(\theta) = 1 + \frac{\mathrm{i}\beta^2}{16}\left(\frac{\sinh\theta}{\cosh\theta + 1/\sqrt{2}} - \frac{\sinh\theta}{\cosh\theta - 1}\right) + \mathrm{O}(\beta^4)\,. \tag{A10}$$

The unwanted non-meromorphic terms exactly cancel out.

Now we consider the heavy particle corresponding to ϕ_2 which are self-conjugate. There are two possible configurations for a type I diagram. One is $b = \phi_1$, the other is $b = \phi_2$ in the notation of figure 2. The ϕ_2 loop contribution is the following:

$$K_2(\theta)^{(I-1)} = \frac{1}{8\sinh\theta}\left(\frac{1}{4\cosh\theta} + \frac{1}{4}\right) \times \left(\frac{-\mathrm{i}}{6}\beta^2\right) \times 12\,. \tag{A11}$$

The ϕ_1 loop contribution is the following:

$$K_2(\theta)^{(I-2)} = \frac{1}{8\sinh\theta}\left(\frac{1}{2\sqrt{4\sinh^2\theta+2}} + \frac{1}{2\sqrt{2}}\right) \times (-\mathrm{i}\beta^2) \times 2\,. \tag{A12}$$

There is no type II diagram for the heavy particle, either.

For a type III diagram, there is a single configuration with $b = \phi_1$, $c = \phi_1$. When $k_1 = 0$, $k_2 = k$,

$$\bar{w}_1 = \sqrt{2} \qquad \tilde{w}_1^+ = 2\cosh\theta + \sqrt{4\sinh^2\theta + 2} \qquad \tilde{w}_1^- = 2\cosh\theta - \sqrt{4\sinh^2\theta + 2} \tag{A13}$$

and when $k_1 = k$, $k_2 = 0$,

$$\bar{w}_1 = \sqrt{4\sinh^2\theta + 2} \qquad \tilde{w}_1^+ = 2\cosh\theta + \sqrt{2} \qquad \tilde{w}_1^- = 2\cosh\theta - \sqrt{2}\,. \tag{A14}$$

The result for a type III diagram can be obtained by inserting the above data into equation (12):

$$K_2(\theta)^{(\mathrm{III})} = \frac{1}{8\sinh\theta}\left(\frac{-\mathrm{i}}{8\sqrt{2}(\sqrt{2}\cosh\theta - 1)} + \frac{\mathrm{i}}{8\sqrt{2}(\sqrt{2}\cosh\theta + 1)} - \frac{\mathrm{i}}{4\sqrt{2}\sqrt{2\sinh^2\theta + 1}}\right) \times(-\beta^2) \times 4\,. \tag{A15}$$

Adding the above contributions as well as the classical value 1, the boundary reflection matrix for the heavy particle is given by

$$K_2(\theta) = 1 + \frac{\mathrm{i}\beta^2}{16}\left(\frac{\sinh\theta}{\cosh\theta} - \frac{\sinh\theta}{\cosh\theta - 1} - \frac{\sinh\theta}{\cosh\theta - 1/\sqrt{2}} + \frac{\sinh\theta}{\cosh\theta + 1/\sqrt{2}}\right) + \mathrm{O}(\beta^4)\,. \tag{A16}$$

The unwanted non-meromorphic terms exactly cancel out once again.

On the other hand, there are two 'minimal' boundary reflection matrices known for the $a_3^{(1)}$ model [3, 5]. None of these agrees with the perturbative result.

We have checked by perturbation theory that this boundary reflection matrix, at one-loop order, satisfies the boundary crossing unitarity relations as well as the boundary bootstrap equations:

$$\begin{aligned} &K_1(\theta)\, K_1(\theta - \mathrm{i}\pi) = S_{11}(2\theta) \qquad K_2(\theta)\, K_2(\theta - \mathrm{i}\pi) = S_{22}(2\theta) \\ &K_2(\theta) = K_1(\theta + \mathrm{i}\pi/4)\, K_1(\theta - \mathrm{i}\pi/4)\, S_{11}(2\theta) \\ &K_1(\theta) = K_3(\theta)\,. \end{aligned} \tag{A17}$$

In one-loop checks, the following identity is useful:

$$\frac{(x + B/2)}{(x)} = 1 + \frac{\mathrm{i}\pi B}{2h}\frac{\sinh\theta}{\cosh\theta - \cos(x\pi/h)} + \mathrm{O}(B^2)\,. \tag{A18}$$

The exact boundary reflection matrix is determined uniquely if we assume the strong–weak coupling 'duality':

$$K_1(\theta) = [1/2][3/2][5/2] \qquad K_2(\theta) = [1/2][3/2]^2[5/2]\,. \tag{A19}$$

On the other hand, the most general solution can be written in the following form under the assumption of the strong–weak coupling 'duality':

$$\begin{aligned} K_1(\theta) &= [1/2]^{a_1}[3/2]^{b_1}[5/2]^{c_1}[7/2]^{d_1} \\ K_2(\theta) &= [1/2]^{a_2}[3/2]^{b_2}[5/2]^{c_2}[7/2]^{d_2}\,. \end{aligned} \tag{A20}$$

Inserting the above into the boundary bootstrap equations, we can obtain linear algebraic relations among the exponents. Solving this system of equations yields

$$\begin{aligned} &a_1 = \text{free} \qquad b_1 = \text{free} \qquad c_1 = b_1 \qquad d_1 = a_1 - 1 \\ &a_2 = -a_1 + b_1 + 1 \qquad b_2 = a_1 + b_1 \qquad c_2 = a_1 + b_1 - 1 \qquad d_2 = -a_1 + b_1\,. \end{aligned} \tag{A21}$$

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